

# SIMPLICIAL COMPLEXES ARE GAME COMPLEXES

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**ABSTRACT.** Strong placement games (SP-games) are a class of combinatorial games whose structure allows one to describe the game via simplicial complexes. A natural question is whether well-known invariants of combinatorial games, such as “game value”, appear as invariants of the simplicial complexes. This paper is the first step in that direction. We show that every simplicial complex encodes a certain type of SP-game (called an “invariant SP-game”) whose ruleset is independent of the board it is played on. We also show that in the class of SP-games isomorphic simplicial complexes correspond to isomorphic game trees, and hence equal game values. We also study a subclass of SP-games corresponding to flag complexes, showing that there is always a game whose corresponding complex is a flag complex no matter which board it is played on.

## 1. INTRODUCTION

The purpose of this paper is to unravel some of the algebraic structure underlying combinatorial games. We show that each simplicial complex is the legal complex of some invariant strong placement game (iSP-game) and board. One implication is that in most situations when studying strong placement games (SP-games) it is enough to consider those with invariance. These results will for example make it easier to study whether each game value under normal play can be achieved by an SP-game, which would affect the study of combinatorial games in general.

In [9] we initiated the idea of using simplicial complexes to algebraically describe SP-games, a class of combinatorial games. To each SP-game we can assign two simplicial complexes, one representing all legal positions, the so called **legal complex**, and one representing the minimal illegal positions, the **illegal complex**. One of the main questions is what complexes appear as game complexes. In Proposition 1.14 we show that every simplicial complex is both a legal and an illegal complex of some SP-game and board. The rulesets of these games can be quite complex though and depend highly on the board on which the game is being played. Thus we introduce **invariance**

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for SP-games, which, in a sense, forces rulesets to be uniform. Invariance is a concept that was introduced for subtraction games (see for example [16], [17]), where it is defined slightly differently due to the different class of games, but has the same intent, namely that the ruleset does not depend on the board. Similar to the previous question, we are interested in which simplicial complexes come from invariant SP-games (iSP games). Lemma 2.9 shows that every simplicial complex without an isolated vertex is the illegal complex of some iSP-game and board, and also that every simplicial complex is a legal complex of an iSP-game. The constructions given in all cases prove the stronger result that such SP-games and boards exist given *any* bipartition of the vertices of the simplicial complex (see Theorems 2.10 and 2.14) into Left and Right positions. This construction then allows us to show that for every SP-game there exists an iSP-game such that their game trees are isomorphic. This in turn implies that their game values are the same under both normal and misère winning conditions. Thus it is enough to only consider iSP-games.

Finally, we discuss another isomorphism for a game  $G$  on a board  $B$  for which the illegal complex is a graph to an **independence game**, a special class of SP-games whose legal complexes are flag complexes. This class includes many games actually played.

In the next two subsections, we give the background in combinatorial game theory and algebra needed for the paper. Please see any of [1, 3, 20] for further information in combinatorial game theory and [12] for the algebra involved. We then show that each simplicial complex is a game complex, and finally in Section 2 we consider invariant SP-games, and independence games in Section 3.

**1.1. Combinatorial Game Theory.** A **combinatorial game** is a 2-player game with perfect information and no chance, where the two players are **Left** and **Right** (denoted by  $L$  and  $R$  respectively) and they do not move simultaneously. For the purposes of this paper, the winning condition is irrelevant as long as it does not contradict the other conditions for the games.

We denote a combinatorial game by its name in SMALL CAPS.

In this paper, a **board** will be a graph. For a game  $G$  played on a board  $B$  we will use the notation  $(G, B)$ . The **pieces**, which can be thought of as tokens or as subgraphs of  $B$ , will be placed on a non-empty collection of vertices - exactly how is given by the **ruleset**. A **position** is a configuration of pieces on the board. A position that can be reached through a sequence of legal moves is called a **legal position**, otherwise we call it an **illegal position**. A **basic position** is a board with only one piece placed.

Given a game  $G$  and a board  $B$ , the **game tree** of  $(G, B)$  is a directed graph tree with the edges labelled  $L$  or  $R$ . The vertices of the tree correspond to positions and  $X \xrightarrow{L} Y$  if there is a legal move for Left from position  $X$  to

$Y$ . Similarly for edges labelled with an  $R$ . The games we consider all have finite game trees.

Brown et al. [5] introduced a subclass of combinatorial games, which they called placement games. Their conditions are slightly weaker than what is required for this work.

**Definition 1.1.** A **strong placement game (SP-game)** is a combinatorial game which satisfies the following:

- (i) The board is empty at the beginning of the game.
- (ii) Players place pieces on empty spaces of the board according to the rules.
- (iii) Pieces are not moved or removed once placed.
- (iv) The rules are such that if it is possible to reach a position through a sequence of legal moves, then any sequence of moves leading to this position consists of legal moves.

Note that condition (iv) in the above definition is necessary for each position to be independent of the order of moves, which results in commutativity when representing positions by monomials (see Section 1.3) and for the hypergraphs representing the game being simplicial complexes.

This condition also implies that any position, whether legal or illegal, in an SP-game can be decomposed into basic positions.

The games in Example 1.2 are examples of SP-games and will be used throughout the document. The first two are games that have been introduced early in the development of game theory (see [3]), but, surprisingly, not much is known about them. As is the case with many games considered in combinatorial game theory, the board is not specified since the games can be played on any graph. In this paper, when we refer to a game we will specify a board.

**Example 1.2.** In SNORT, players place a piece on a single vertex which is not adjacent to a vertex containing a piece from their opponent.

In COL, players place a piece on a single vertex which is not adjacent to a vertex containing one of their own pieces.

In NOGO, players place a piece on a single unoccupied vertex. At every point in the game, for each maximal group of connected vertices of the board that contain pieces placed by the same player, one of these needs to be adjacent to an empty vertex.

In DOMINEERING (see [2] and [15]), which is played on grids, both players place dominoes. Left may only place vertically, and Right only horizontally. The vertices of the board are the squares of the grid, and each piece occupies two vertices.

In PARTIZAN OCTAL games, the board is a strip and the players have ‘dominoes’ of different lengths [11, 18].

Other examples of SP-games are NODE-KAYLES and ARC-KAYLES (see for example [4], [10], [19]).

Depending on a fixed winning condition, combinatorial games can be divided into equivalence classes. The simplest game (essentially smallest game tree) is called the **game value** of that equivalence class. Game values form a partially ordered semi-group. When one game consists of two sub-games on different boards, then to find the game value it is sufficient to calculate the game values of the sub-games and taking advantage of the additive structure. This is a very useful concept in combinatorial game theory. For more details see [20].

**1.2. Combinatorial Commutative Algebra.** Simplicial complexes are one of the main constructs we use to study SP-games. We begin by introducing the required concepts.

**Definition 1.3.** An (abstract) **simplicial complex**  $\Delta$  on a finite vertex set  $V$  is a set of subsets (called **faces**) of  $V$  with the conditions that if  $A \in \Delta$  and  $B \subseteq A$ , then  $B \in \Delta$ . The **facets** of a simplicial complex  $\Delta$  are the maximal faces of  $\Delta$  with respect to inclusion. A **non-face** of a simplicial complex  $\Delta$  is a subset of its vertices that is not a face.

Note that a simplicial complex is uniquely determined by its facets. Thus a simplicial complex  $\Delta$  with facets  $F_1, \dots, F_k$  is denoted by  $\Delta = \langle F_1, \dots, F_k \rangle$ .

A simplicial complex of the form  $\Delta = \langle \{i_1, i_2, \dots, i_r\} \rangle$ , where  $\{i_1, i_2, \dots, i_r\}$  is the vertex set of  $\Delta$ , is called a **simplex**.

**Definition 1.4.** Given a face  $F$  of a simplicial complex  $\Delta$ , its **dimension**  $\dim(F)$  is  $|F| - 1$ . The dimension of the simplicial complex  $\Delta$  is the maximum dimension of any of its faces. A simplicial complex  $\Delta$  is called **pure** if all its facets are of the same dimension. The **k-skeleton**  $\Delta^{[k]}$  of a simplicial complex  $\Delta$  is the simplicial complex whose facets are the  $k$ -dimensional faces of  $\Delta$ .

The other structures used to study SP-games are square-free monomial ideals, which we introduce now.

**Definition 1.5.** Let  $k$  be a field and  $R$  the polynomial ring  $k[x_1, \dots, x_n]$ . A product  $x_1^{a_1} \dots x_n^{a_n} \in R$ , where the  $a_i$  are non-negative integers, is called a **monomial**. Such a monomial is called **square-free** if each  $a_i$  is either 0 or 1.

**Definition 1.6.** Let  $k$  be a field and  $R$  the polynomial ring  $k[x_1, \dots, x_n]$ . A **monomial ideal** of  $R$  is an ideal generated by monomials in  $R$ . A monomial ideal is called a **square-free monomial ideal** if it is generated by square-free monomials.

Let  $k$  be a field and  $R = k[x_1, \dots, x_n]$  a polynomial ring. There is a one-to-one correspondence between subsets  $\{i_1, \dots, i_r\}$  of  $[n]$  and square-free monomials  $x_{i_1} \dots x_{i_r}$  of  $R$ . Using this observation we can associate to a square-free monomial ideal two unique simplicial complexes: the facet complex and the Stanley-Reisner complex.

**Definition 1.7.** The **facet complex** of a square-free monomial ideal  $I$  of  $R$ , denoted by  $\mathcal{F}(I)$ , is the simplicial complex whose facets correspond to the square-free monomials in the minimal generating set of  $I$ . The **Stanley-Reisner complex** of a square-free monomial ideal  $I$  of  $R$ , denoted by  $\mathcal{N}(I)$ , is the simplicial complex whose faces correspond to the square-free monomials not in  $I$ . In other words,

$$\begin{aligned}\mathcal{F}(I) &= \langle \{i_1, \dots, i_r\} \mid x_{i_1} \cdots x_{i_r} \text{ minimal generator of } I \rangle \text{ and} \\ \mathcal{N}(\Delta) &= \langle \{i_1, \dots, i_r\} \mid x_{i_1} \cdots x_{i_r} \notin I \rangle.\end{aligned}$$

This correspondence works in the opposite direction as well.

**Definition 1.8.** The **facet ideal** of a simplicial complex  $\Delta$ , denoted by  $\mathcal{F}(\Delta)$ , is the ideal of  $R$  generated by the monomials corresponding to the facets of  $\Delta$ . The **Stanley-Reisner ideal** of a simplicial complex  $\Delta$ , denoted by  $\mathcal{N}(\Delta)$ , is the ideal of  $R$  generated by the monomials corresponding to the minimal non-faces of  $\Delta$ . In other words,

$$\begin{aligned}\mathcal{F}(\Delta) &= (x_{i_1} \cdots x_{i_r} \mid \{i_1, \dots, i_r\} \text{ facet of } \Delta) \text{ and} \\ \mathcal{N}(\Delta) &= (x_{i_1} \cdots x_{i_r} \mid \{i_1, \dots, i_r\} \notin \Delta).\end{aligned}$$

**1.3. Game Complexes and Ideals.** We now introduce the construction of simplicial complexes and square-free monomial ideals which are related to SP-games. Unless otherwise specified, let the underlying ring be  $R = k[x_1, \dots, x_m, y_1, \dots, y_n]$ , where  $k$  is a field,  $m$  the number of basic positions with a Left piece, and  $n$  the number of basic positions with a Right piece.

A square-free monomial  $z$  of  $R$  represents a position  $P$  in the game if it is the product over those  $x_i$  and  $y_j$  such that Left has played in the basic position  $i$  and Right has played in the basic position  $j$  in order to reach  $P$ . By condition (iv) in Definition 1.1, the order of moves to reach  $P$  does not matter, thus we have commutativity.

**Example 1.9.** Consider DOMINEERING played on the board  $B$  given in Fig. 1.

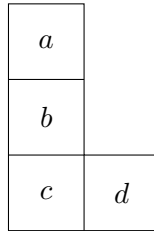


FIGURE 1. Board  $B$  with squares labelled

Since Left and Right both play dominoes, the basic positions are to place a domino on vertices  $a, b$  (basic position 1), on  $b, c$  (basic position 2), or

on  $c, d$  (basic position 3). Thus the underlying ring in this case is  $R = k[x_1, x_2, x_3, y_1, y_2, y_3]$ .

Since Left may only place a domino vertically, the basic position represented by  $x_1$  and  $x_2$  are legal, while  $x_3$  is illegal. Similarly, for Right  $y_1$  and  $y_2$  are illegal, while  $y_3$  is legal.

The monomial  $x_1y_3$  represents the position in which Left has placed a domino on vertices  $a$  and  $b$ , and Right has played on  $c$  and  $d$ , which is a legal position. Similarly,  $x_2y_3$  represents the position where Left has played on vertices  $b$  and  $c$ , while Right has played on  $c$  and  $d$ , which is illegal since the two dominoes overlap.

A legal position is called a **maximal legal position** if placing any further piece is illegal, i.e. it is not properly contained in any other legal position.

If we sort the monomials representing illegal positions by divisibility, the positions corresponding to the minimal elements are called **minimal illegal positions**. Equivalently, an illegal position is a minimal illegal position if any proper subset of the pieces placed forms a legal position.

**Definition 1.10.** [9] If  $(G, B)$  is an SP-game, then

- The **legal ideal**,  $\mathcal{L}_{G,B}$ , is the ideal of  $R$  generated by the monomials representing maximal legal positions.
- The **illegal ideal**,  $\mathcal{ILL}_{G,B}$ , is the ideal generated by the monomials representing minimal illegal positions.
- The **legal complex**,  $\Delta_{G,B}$ , is the facet complex of the legal ideal.
- The **illegal complex**,  $\Gamma_{G,B}$ , is the facet complex of the illegal ideal.

If a given simplicial complex is the legal or illegal complex of some game and board, we also call it a **game complex**.

*Remark 1.11.* Note that the faces of the legal complex  $\Delta_{G,B}$  represent the legal positions of  $(G, B)$ , while the facets of  $\Gamma_{G,B}$  represent the minimal illegal positions. In short we have

- (1)  $\mathcal{L}_{G,B} = \mathcal{F}(\Delta_{G,B})$ ,
- (2)  $\mathcal{ILL}_{G,B} = \mathcal{F}(\Gamma_{G,B}) = \mathcal{N}(\Delta_{G,B})$ ,

or equivalently

- (1)  $\Delta_{G,B} = \mathcal{F}(\mathcal{L}_{G,B}) = \mathcal{N}(\mathcal{ILL}_{G,B})$ ,
- (2)  $\Gamma_{G,B} = \mathcal{F}(\mathcal{ILL}_{G,B})$ .

This will be used throughout this paper.

Note that condition (iv) in Definition 1.1 implies that the order of moves does not matter, which gives us commutativity when representing positions by monomials, thus the legal and illegal ideal are indeed commutative ideals. The condition also implies that given any legal position, any subset of the pieces played gives a legal position as well, and thus the hypergraphs representing the game are indeed simplicial complexes.

We will continue Example 1.9 to demonstrate these concepts. This also illustrates again that the vertices of the complexes are the basic positions, not the vertices of the graph/board.

**Example 1.12.** Consider DOMINEERING played on the board  $B$  given in Fig. 1. Our underlying ring is  $R = k[x_1, x_2, x_3, y_1, y_2, y_3]$ .

The maximal legal positions are represented by the monomials  $x_1y_3$  and  $x_2$ . Thus we have the legal ideal

$$\mathcal{L}_{\text{DOMINEERING}, B} = \langle x_1y_3, x_2 \rangle.$$

The legal complex is given in Fig. 2.

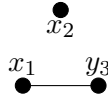


FIGURE 2. The legal complex  $\Delta_{\text{DOMINEERING}, B}$

The minimal illegal positions are represented by the monomials  $x_1x_2$ ,  $x_2y_3$ ,  $x_3$ ,  $y_1$ , and  $y_2$ . Thus we have the illegal ideal

$$\mathcal{ILL}_{\text{DOMINEERING}, B} = \langle x_1x_2, x_2y_3, x_3, y_1, y_2 \rangle.$$

The illegal complex is given in Fig. 3.

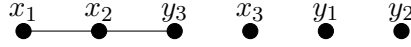


FIGURE 3. The illegal complex  $\Gamma_{\text{DOMINEERING}, B}$

It is important to note that the legal and illegal complexes and corresponding ideals have an extra layer of structure. The monomials have elements  $\{x_1, x_2, \dots, x_n\}$  and  $\{y_1, y_2, \dots, y_n\}$  and the complexes have their elements partitioned into those corresponding to the Left and Right basic positions. Isomorphisms between game complexes and ideals must also preserve these partitions. Thus, when showing a complex is isomorphic to a game complex, we must also specify the partition.

In general, we call a simplicial complex whose vertex set is bipartioned into sets  $\mathfrak{L}$  and  $\mathfrak{R}$  an  $(\mathfrak{L}, \mathfrak{R})$ -labelled simplicial complex.

The following proposition shows that two games with isomorphic legal complexes have isomorphic game trees, and as a consequence the same game value under most winning conditions (such as normal play and misère, see [20]).

**Proposition 1.13 (Isomorphic Game Trees of SP-Games).** *Two SP-games  $G_1$  and  $G_2$  played on boards  $B_1$  and  $B_2$  respectively have isomorphic legal complexes if and only if the game trees of  $G_1$  and  $G_2$  are isomorphic.*

*Proof.* We prove that isomorphic legal complexes imply isomorphic game trees by induction on the size of the faces (i.e. the number of pieces in a position). The empty face (i.e. empty board) corresponds to the root of the game tree, thus is trivially the same for both games.

Now assume that the game trees are isomorphic up to positions with  $k$  pieces played.

Consider a position  $P_1$  in the game  $G_1$  played on  $B_1$  with  $k$  pieces played. Let  $F_1$  be the face of  $\Delta_{G_1, B_1}$  (of dimension  $k - 1$ ) corresponding to  $P_1$ . Since  $\Delta_{G_1, B_1}$  and  $\Delta_{G_2, B_2}$  are isomorphic, there exists a face  $F_2 \in \Delta_{G_2, B_2}$  (of dimension  $k - 1$ ) isomorphic to  $F_1$ , corresponding to a position  $P_2$  of  $G_2$ , which also has  $k$  pieces placed.

Now let  $P'_1$  be any option of  $P_1$  and  $F'_1$  be the corresponding face in  $\Delta_{G_1, B_1}$ . Then there exists a vertex  $v$  such that  $F'_1 = F_1 \cup \{v\}$ . Let  $F'_2$  be the face of  $\Delta_{G_2, B_2}$  isomorphic to  $F'_1$ . Then there exists a vertex  $w$  (corresponding to  $v$ ) such that  $F'_2 = F_2 \cup \{w\}$ . Thus the position  $P'_2$  corresponding to  $F'_2$  is an option of  $P_2$ .

Further, since the legal complexes have the same bipartition, we have that the following are equivalent:

- (1) The position  $P'_1$  is a Left- (Right-)option of  $P_1$ .
- (2) The vertex  $v$  belongs to  $\mathfrak{L}(\mathfrak{R})$ .
- (3) The vertex  $w$  belongs to  $\mathfrak{L}(\mathfrak{R})$ .
- (4) The position  $P'_2$  is a Left- (Right-)option of  $P_2$ .

Thus for any option of  $P_1$  there exists an option of  $P_2$  and vice-versa, which shows that the game trees of  $G_1, B_1$  and  $G_2, B_2$  are isomorphic up to positions of  $k + 1$ , and by induction they are entirely isomorphic.

The proof of the converse is similar and so we omit it.  $\square$

If the illegal complexes are isomorphic, it is not always true that the game trees are isomorphic. For example, consider a game  $G$  in which neither player can place on a vertex of degree 1. We then have

$$\Gamma_{G, P_2} = \langle x_1, x_2, y_1, y_2 \rangle \cong \Gamma_{G, P_3} = \langle x_1, x_3, y_1, y_3 \rangle.$$

The legal complexes  $\Delta_{G, P_2} = \emptyset$  and  $\Delta_{G, P_3} = \langle x_2, y_2 \rangle$  are not isomorphic, and thus by Proposition 1.13 their game trees are not either. Another occurrence of this is if there are moves that are always playable in one game, but these moves do not occur at all in the second game. This problem appears and is dealt with in the proof of Theorem 2.14.

A natural and important question is whether any given simplicial complex  $\Delta$  is the legal or illegal complex of some game. We will answer this question positively in both cases. This will allow us to view properties of games as properties of simplicial complexes and vice-versa. We are able to show this for any bipartition of the vertices into Left  $\mathfrak{L}$  and Right  $\mathfrak{R}$ , where  $\mathfrak{L}$  or  $\mathfrak{R}$  could even be the empty set.



**Proposition 1.14 (Games from Simplicial Complexes).** *Given an  $(\mathfrak{L}, \mathfrak{R})$ -labelled simplicial complex  $\Delta$ , there exist SP-games  $G_1, G_2$  and a board  $B$  such that*

- (a)  $\Delta = \Delta_{G_1, B}$  and
- (b)  $\Delta = \Gamma_{G_2, B}$

*and the sets of Left (respectively Right) positions is  $\mathfrak{L}$  (respectively  $\mathfrak{R}$ ).*

*Proof.* Let  $m = |\mathfrak{L}|$  and  $n = |\mathfrak{R}|$ . Let  $B$  be the board consisting of  $m$  disjoint 3-cycles and  $n$  disjoint 4-cycles. In the games  $G_1$  and  $G_2$ , Left will be playing 3-cycles, while Right will be playing 4-cycles.

In  $\Delta$ , label the vertices belonging to  $\mathfrak{L}$  as  $1, \dots, m$ , and the vertices in  $\mathfrak{R}$  as  $m+1, \dots, n+m$ . Similarly, label the 3-cycles of  $B$  as  $1, \dots, m$ , and the 4-cycles as  $m+1, \dots, n+m$ .

(a) In  $G_1$ , playing on a set of cycles of  $B$  is legal if and only if the corresponding set of vertices in  $\Delta$  forms a face.

(b) In  $G_2$ , playing on a set of cycles of  $B$  is legal if and only if the corresponding set of vertices in  $\Delta$  does not contain a facet.

It is now easy to see that  $\Delta = \Delta_{G_1, B}$  and  $\Delta = \Gamma_{G_2, B}$ .  $\square$

As seen above, it is rather simple to construct games on fixed boards from simplicial complexes by restricting the legal moves to certain parts of the board. We now move on to look at games where such restrictions can be relaxed. We call these invariant games.

## 2. INVARIANT GAMES

As we have shown in the previous section, every simplicial complex is the legal or illegal complex of some SP-game and board. The rules created as part of this construction, however, depend heavily on the board. We now define the concept of invariance for SP-games, which in a sense forces the ruleset to be “uniform” across the board.

**Definition 2.1.** The ruleset of an SP-game is **invariant** if the following conditions hold:

- Every basic position is legal.
- The ruleset does not depend on the board, i.e. if  $B_1$  and  $B_2$  are isomorphic subgraphs of any board  $B$ , then a position on  $B_1$  is legal if and only if its isomorphic image on  $B_2$  is legal.

If the ruleset of an SP-game is invariant, we also say that the game is an **invariant strong placement game (iSP-game)**.

COL and SNORT are examples of games that are invariant, while DOMINEERING and NOGO are not. In DOMINEERING half of the basic positions are illegal (Right cannot play vertically, while Left cannot play horizontally). That NOGO is not invariant is not as obvious. Indeed on most boards both conditions hold, but whenever the board has an isolated vertex, playing on it is illegal (thus the basic position corresponding to that vertex is illegal).

Similar to the question of the previous section, we are interested in which simplicial complexes appear as the legal or illegal complex of an iSP-game.

We will show below that the illegal complex of an iSP-game cannot contain an isolated vertex.

**Proposition 2.2.** *Let  $\Gamma$  be a simplicial complex. If  $\Gamma$  is the illegal complex of some iSP-game and some board then  $\Gamma$  has no facets that are one-element sets.*

*Proof.* Assume that  $\Gamma$  has a facet that is a one-element set, i.e. an isolated vertex, and label this vertex  $a$ . If  $\Gamma$  is the illegal complex of some SP-game  $G$  and board  $B$ , then since  $\{a\}$  is a facet of  $\Gamma$ , there exists a basic position (corresponding to the vertex  $a$ ) which is illegal. Thus  $G$  does not satisfy the first condition of invariance.  $\square$

Other than the isolated vertex situation, there is no obstruction for a simplicial complex  $\Gamma$  being an illegal complex. We set out to prove this by constructing a  $\Gamma$ -board and a  $\Gamma$ -game.

**Construction 2.3 ( $\Gamma$ -Board).** Given an  $(\mathfrak{L}, \mathfrak{R})$ -labelled simplicial complex  $\Gamma$  with no isolated vertices we can construct a graph  $B_\Gamma$  (called the  $\Gamma$ -Board) as follows:

If  $\Gamma$  is empty, then let  $B_\Gamma$  be empty.

If  $\Gamma$  is non-empty, then let  $H = \Gamma^{[1]}$ , i.e. the underlying graph of  $\Gamma$ . Let  $n$  be the number of vertices in the graph  $H$  and (re)label the vertices of  $H$  as  $1, \dots, n$ . Begin constructing the board  $B_\Gamma$  by using  $n$  cycles of sizes  $n^4 + 4$  and  $n^4 + 5$  and label these  $1, \dots, n$  so that cycle  $i$  will have size  $n^4 + 4$  if the vertex  $i$  in  $H$  belongs to  $\mathfrak{L}$ , and size  $n^4 + 5$  if the vertex  $i$  belongs to  $\mathfrak{R}$ . For each cycle, designate  $n - 1$  consecutive vertices for joining, called **connection vertices** (see Fig. 4).

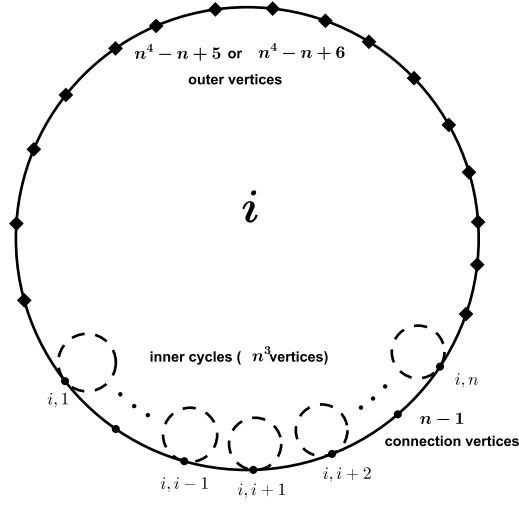
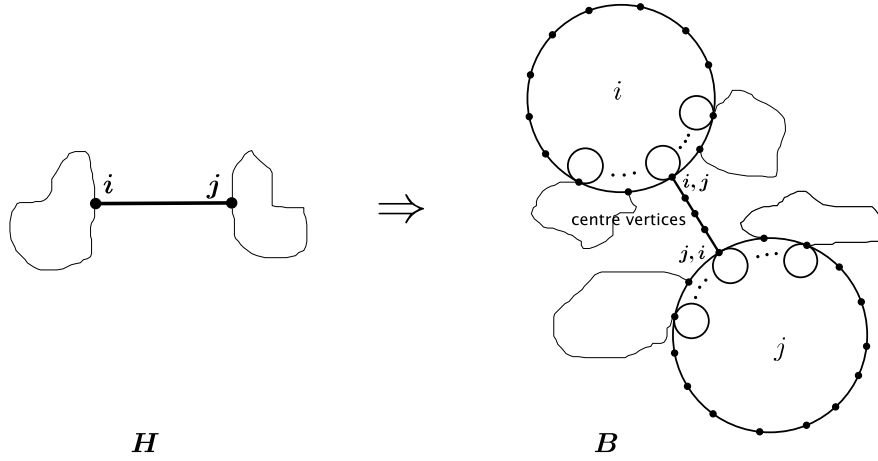
Call the remaining vertices **outer vertices**. To each connection vertex, join a cycle of length  $n^3$  (called **inner cycles**). In cycle  $i$  label the connection vertices as  $i, j$  where  $j = 1, \dots, n$  and  $j \neq i$ .

Label the edges in  $H$  as  $1, \dots, k$ . If the endpoints of the edge  $l$  are the vertices  $i$  and  $j$ , then add a path of  $2 + l$  vertices to  $B_\Gamma$ , whose end vertices are  $i, j$  and  $j, i$  (see Fig. 5). The  $l$  vertices between  $i, j$  and  $j, i$  are called **centre vertices**.

As an example for this construction, consider the following:

**Example 2.4.** Let  $\Gamma$  be a path of three vertices so that  $H = \Gamma$ . Let the two end vertices belong to  $\mathfrak{L}$ , and the centre vertex to  $\mathfrak{R}$ . Since  $\Gamma$  consists of three vertices, i.e.  $n = 3$ , the cycle  $i$  (where  $i \in \mathfrak{L}$ ) is of length  $3^4 + 4 = 85$  with two cycles of length  $3^3 = 27$  joined to two adjacent vertices, and the cycle  $j$  (where  $j \in \mathfrak{R}$ ) is of length 86 with two cycles of length 27 joined to two adjacent vertices.

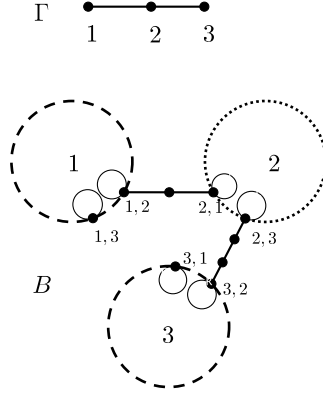
Label the edge between vertex 1 (an end vertex) and vertex 2 (the centre vertex) as 1, and the edge between vertex 2 and vertex 3 (the other end vertex) as 2.

FIGURE 4. Cycle  $i$  in the board  $B_\Gamma$ FIGURE 5. Effect of an edge in  $H$  on the board  $B_\Gamma$ 

The board  $B_\Gamma$  is given in Fig. 6. Dashed cycles consist of 85 vertices, and dotted cycles of 86 vertices, with the two labelled vertices adjacent in both cases. The smaller solid cycles consist of 27 vertices.

For the next construction, we will have to specify what is meant with distance between pieces.

**Definition 2.5.** Let two pieces  $P_1$  and  $P_2$  be placed on a board  $B$  and let  $V_1$  and  $V_2$  be the set of vertices on which  $P_1$ , respectively  $P_2$ , was placed.

FIGURE 6. Constructing  $B_{P_3}$ 

We then define the **distance**  $d(P_1, P_2)$  between  $P_1$  and  $P_2$  by

$$d(P_1, P_2) = \min\{d(v_1, v_2) : v_1 \in V_1, v_2 \in V_2\},$$

where  $d(v_1, v_2)$  is the graph theoretic distance between  $v_1$  and  $v_2$ , i.e. the minimum number of edges of a path in  $B$  with endpoints  $v_1$  and  $v_2$ .

**Construction 2.6 ( $\Gamma$ -Game).** Given an  $(\mathfrak{L}, \mathfrak{R})$ -labelled simplicial complex  $\Gamma$  with no isolated vertices we construct an SP-game  $G_\Gamma$  (called the  $\Gamma$ -Game).

If  $\Gamma$  is empty, then let  $G_\Gamma$  be the game in which Left and Right place pieces on a single vertex with no restrictions.

If  $\Gamma$  is non-empty, then construct  $G_\Gamma$  as follows:

- Let  $n$  be the number of vertices of  $\Gamma$ . Label the edges (the 1-dimensional faces) of  $\Gamma$  as  $\{1, \dots, k\}$ .
- Left plays cycles of length  $n^4 + 4$  with cycles of length  $n^3$  joined to  $n - 1$  consecutive vertices,
- Right plays cycles of length  $n^4 + 5$  with cycles of length  $n^3$  joined to  $n - 1$  consecutive vertices (i.e. the pieces are as the structure given in Fig. 4), and
- Let  $F$  be a facet of  $\Gamma$  of dimension  $f - 1$ , whose 1-dimensional faces are labelled  $k_1, \dots, k_l$ , where  $l = \binom{f}{2}$ . We call the set  $\{k_1 + 1, \dots, k_l + 1\}$  the **id-set** of  $F$ . Then no sets of  $f$  pieces are allowed such that the set of distances between any two pieces is exactly the id-set of  $F$ .

**Example 2.7.** Let  $\Gamma$  be a path of three vertices so that  $n = 3$ . Left's pieces are cycles of length  $3^4 + 4 = 85$  with two cycles of length  $3^3 = 27$  joined to two adjacent vertices, and Rights pieces are cycles of length 86 with two cycles of length 27 joined to two adjacent vertices.

Since the facets of  $\Gamma$  are the two edges (thus of size 2), we have that no two pieces in  $G_\Gamma$  are allowed to have distance 2 or distance 3, i.e. the id-sets are  $\{2\}$  and  $\{3\}$ .

**Example 2.8.** Consider  $\Gamma = \langle abc, ad \rangle$ . Label the edge between  $a$  and  $b$  as 1, between  $b$  and  $c$  as 2, between  $c$  and  $a$  as 3, and between  $a$  and  $d$  as 4.

For the facet  $abc$  we have the id-set  $\{1 + 1, 2 + 1, 3 + 1\} = \{2, 3, 4\}$ . Thus in the  $\Gamma$ -game  $G_\Gamma$  we cannot have three pieces where the distances between pairs are  $\{2, 3, 4\}$ , while two with any one of these distance are allowed.

For the facet  $ad$  we have the id-set  $\{4 + 1\} = \{5\}$ . Thus in  $G_\Gamma$  we cannot have any two pieces with distance 5.

**Lemma 2.9.** *Given an  $(\mathfrak{L}, \mathfrak{R})$ -labelled simplicial complex  $\Gamma$  with no isolated vertices, the  $\Gamma$ -game  $G_\Gamma$  is an iSP-game.*

*Proof.* If  $\Gamma$  is empty, then  $G_\Gamma$  has no illegal positions, thus is trivially invariant.

If  $\Gamma$  is non-empty, then since  $\Gamma$  has no isolated vertices, all facets have at least one edge and therefore all id-sets are non-empty. In particular, this means that every illegal position of  $G_\Gamma$  has at least two pieces, so there are no illegal basic positions.

Now suppose that we are playing  $G_\Gamma$  on a board  $B$ , and let  $B_1$  and  $B_2$  be isomorphic subgraphs of  $B$ . A position  $P$  is legal on  $B_1$  if and only if there is no id-set which is contained in the set of distances between pieces of  $P$ , which holds if and only if  $P$  is legal on  $B_2$ .

Thus  $G_\Gamma$  is invariant.  $\square$

The following statement will prove that every simplicial complex without isolated vertices can appear as the illegal complex of (many!) iSP-games.

**Theorem 2.10 (Invariant Game from Illegal Complex).** *Given an  $(\mathfrak{L}, \mathfrak{R})$ -labelled simplicial complex  $\Gamma$  with no isolated vertices, fix labellings of the vertices and of the edges. Then  $\Gamma$  is the illegal complex of the  $\Gamma$ -game  $G_\Gamma$  played on the  $\Gamma$ -board  $B_\Gamma$ , i.e.  $\Gamma_{G_\Gamma, B_\Gamma} = \Gamma$ .*

*Proof.* Let  $B = B_\Gamma$  and  $G = G_\Gamma$  be the  $\Gamma$ -board and  $\Gamma$ -game respectively, with the same labelling of the edges of  $\Gamma$  if  $\Gamma$  is nonempty.

If  $\Gamma$  is empty, then  $G_\Gamma$  has no illegal positions, thus  $\Gamma_{G, B}$  is also empty.

To show that indeed  $\Gamma_{G, B} = \Gamma$  for  $\Gamma$  nonempty, we will begin by showing that their vertex sets have the same size.

Let  $H = \Gamma^{[1]}$ . Clearly Left can place one of her pieces on the cycle labelled  $i$  in  $B$  if the vertex  $i$  of  $H$  belongs to  $\mathfrak{L}$ . Similarly Right can place on cycles labelled  $j$  where  $j \in \mathfrak{R}$ . Thus each vertex in  $H$  corresponds to a position in the game  $G$  played on  $B$ .

We now need to show that there are no other ways for Left or Right to place pieces than what was previously mentioned, i.e. that the positions of  $G$  played on  $B$  correspond exactly to the vertices of  $H$ .

Let  $n$  be the number of vertices of  $H$  and  $k$  be the number of edges. The cycles in  $B$  which only use connection and centre vertices have size at most

$n(n-1) + \frac{k(k+1)}{2}$  (there are  $n(n-1)$  connection vertices and  $1 + \dots + k$  centre vertices). Since there are at most  $\binom{n}{2}$  edges in  $H$ , we have

$$\begin{aligned} n(n+1) + \frac{k(k+1)}{2} &\leq n(n+1) + \frac{\frac{n(n+1)}{2} \left( \frac{n(n+1)}{2} + 1 \right)}{2} \\ &= \frac{1}{8}n^4 + \frac{1}{4}n^3 + \frac{11}{8}n^2 + \frac{5}{4}n \end{aligned}$$

which is less than  $n^4 + 4$  for all whole numbers.

Thus such cycles are shorter than  $n^4 + 4$ , and Left and Right will not be able to play on those.

Furthermore, any cycle of length  $n^4 + 4$  or  $n^4 + 5$  in  $B$  needs to include the outer vertices of some cycle  $i$  (since as above cycles using only connection and centre vertices are shorter, and the inner cycles are shorter). To then construct a cycle of that length without using all connection vertices of cycle  $i$ , the cycle would have to include at least one centre vertex. Since centre vertices do not have cycles of length  $n^3$  added, this implies that neither Left or Right could play there.

Thus Left and Right are only able to play on the labelled cycles.

Further, since the pieces consist of cycles with a differing number of vertices, either player will only be able to play on the cycles of  $B$  that are designated to them. Thus there are  $n$  positions, in each of which only one player can play, all corresponding to vertices of  $\Gamma$ . The vertices of  $\Gamma_{G,B}$  are thus a subset of the vertices of  $\Gamma$  and  $\Gamma_{G,B}$  has less vertices than  $\Gamma$  if and only if there exists at least one position in which it is never illegal to play, which we will show cannot happen as part of the rest of the proof.

It remains to show that the facets of  $\Gamma_{G,B}$  and  $\Gamma$  correspond.

Consider a facet consisting of the vertices  $i_1, \dots, i_k$  in  $\Gamma$ , thus any two vertices have an edge between them in  $H$ , and let these edges be  $j_1, \dots, j_l$ . Then the positions  $i_a$  and  $i_b$ ,  $a, b \in \{1, \dots, k\}$ , in  $B$  have distance  $j_c + 1$ , where  $j_c$  is the edge between  $i_a$  and  $i_b$  in  $H$ , (since we joined a path of length  $j_c + 2$  to their connection vertices). Thus it is illegal to play in all  $k$  positions (and this is a minimal illegal position), and thus there is a facet consisting of the vertices  $i_1, \dots, i_k$  in  $\Gamma_{G,B}$ .

Now let the vertices  $i_1, \dots, i_k$  form a facet in  $\Gamma_{G,B}$ . Assume that  $i_1, \dots, i_k$  do not form a facet in  $\Gamma$ . If some subset  $S$  of these vertices forms a facet, then by construction of  $G$  it would be illegal to play pieces on all of the cycles in  $B$  corresponding to vertices in  $S$ . Thus  $i_1, \dots, i_k$  is not a *minimal* illegal position, a contradiction to those vertices forming a facet in  $\Gamma_{G,B}$ . If on the other hand  $i_1, \dots, i_k$  is strictly contained in some facet  $F$  of  $\Gamma$ , then by construction of  $G$  it is legal to play on cycles  $i_1, \dots, i_k$  in  $B$ . Thus  $i_1, \dots, i_k$  is not an *illegal* position, a contradiction to those vertices forming a facet in  $\Gamma_{G,B}$ . Therefore  $i_1, \dots, i_k$  is a facet of  $\Gamma$ .

Finally, since  $H$  has no isolated vertices (by  $\Gamma$  not having such), the vertex set of  $\Gamma_{G,B}$  is a subset of the vertex set of  $H$ , i.e. the vertex set of  $\Gamma$ . Since

furthermore the facets of  $\Gamma_{G,B}$  and  $\Gamma$  correspond, we have that the vertex set of  $\Gamma_{G,B}$  is equal to that of  $\Gamma$ .

Consequently, the simplicial complexes  $\Gamma$  and  $\Gamma_{G,B}$  have the same vertex and facet sets, which proves  $\Gamma = \Gamma_{G,B}$ .  $\square$

**Example 2.11.** Let  $\Gamma$  be a path of three vertices. Let  $B = B_\Gamma$  (see Example 2.4) and  $G = G_\Gamma$  (see Example 2.7).

Then  $\Gamma_{G,B} = \Gamma$ .

Note: Simpler constructions with smaller cycles and pieces are often possible (as shown in the next example), but the above construction is guaranteed to work.

**Example 2.12.** Let  $\Gamma$  be as in Example 2.11. Let Left play cycles of length 3, and Right cycles of length 4. For the board  $B'$  given in Fig. 7, it is easy to check that  $\Gamma_{G',B'} = \Gamma$ , where  $G'$  is the SP-game with no additional rules (this in particular means that pieces are not allowed to overlap).

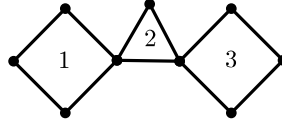


FIGURE 7. Smaller board  $B'$

The following theorem summarizes our results about illegal complexes of iSP-games.

**Theorem 2.13.** *A given simplicial complex  $\Gamma$  is the illegal complex of some iSP-game  $G$  played on a board  $B$  if and only if  $\Gamma$  has no isolated vertices.*

*Proof.* By Proposition 2.2 we have that if  $\Gamma$  is the illegal complex of an iSP-game, then  $\Gamma$  has no isolated vertices.

Conversely, if  $\Gamma$  has no isolated vertices, then by Theorem 2.10, we have that  $\Gamma$  is the illegal complex of some iSP-game and board.  $\square$

We will now consider legal complexes. The first result shows that *every* simplicial complex is the legal complex of some iSP-game and board:

**Theorem 2.14 (Invariant Game from Legal Complex).** *Given any  $(\mathfrak{L}, \mathfrak{R})$ -labelled simplicial complex  $\Delta$ , we can construct an iSP-game  $G$  and a board  $B$  such that  $\Delta = \Delta_{G,B}$  and the sets of Left, respectively Right, positions is  $\mathfrak{L}$ , respectively  $\mathfrak{R}$ .*

*Proof.* We will prove this separately for the case in which the simplicial complex  $\Delta$  is not a simplex (corresponding to  $\mathcal{F}(\mathcal{N}(\Delta))$  has at least one 1-dimensional face), and when it is a simplex (corresponding to  $\mathcal{F}(\mathcal{N}(\Delta))$  has only isolated vertices or is empty).

*Case 1:* If  $\Delta$  is not a simplex, the construction is as follows:

Given  $\Delta$ , let  $\Gamma = \mathcal{F}(\mathcal{N}(\Delta))$ , i.e. the simplicial complex whose facets correspond to the minimal non-faces of  $\Delta$ .

Let  $i$  be a vertex in  $\Delta$ . If  $\Delta$  has at least one facet that does not contain  $i$ , then  $i$  will also be a vertex of  $\Gamma$ . Otherwise it is not a vertex of  $\Gamma$ .

Let the vertex set of  $\Gamma$  be bipartitioned into  $\mathfrak{L}$  and  $\mathfrak{R}$  the same way that the vertex set of  $\Delta$  is. Let  $n$  be the number of vertices in  $\Gamma$  and let  $G$  be the  $\Gamma$ -game and  $B_0$  be the  $\Gamma$ -board, so that  $\Gamma_{G,B_0} = \Gamma$ . If  $\Delta$  has a vertex that is contained in every facet, then the underlying rings of  $\Gamma_{G,B_0}$  and  $\Gamma$  are not the same, and we thus have to adjust the board as follows:

Without loss of generality, let  $1, \dots, k$  be the vertices of  $\Delta$  that are contained in every facet. Then for  $l = 1, \dots, k$  let  $B_l = B_{l-1} \cup C^l$  where  $C^l$  is a cycle of length  $n^4 + 4$  (if the vertex  $l$  belongs to  $\mathfrak{L}$ ) or length  $n^4 + 5$  (if it belongs to  $\mathfrak{R}$ ) with  $n - 1$  cycles of length  $n^3$  joined to  $n - 1$  consecutive vertices. Let  $B = B_k$ . When playing the game  $G$  on  $B$ , it is always legal to play on the disjoint  $C^l$  for either Left or Right, thus these positions are never part of a minimal illegal position, which shows that  $\Gamma_{G,B_0} = \Gamma_{G,B}$ . Furthermore, the underlying rings of  $\Gamma$  and  $\Gamma_{G,B}$  are the same.

It immediately follows that

$$\Delta_{G,B} = \mathcal{N}(\mathcal{F}(\Gamma_{G,B})) = \mathcal{N}(\mathcal{F}(\Gamma)) = \Delta.$$

*Case 2:* If  $\Delta$  is a simplex, we can construct  $G$  and  $B$  as follows:

Let  $n$  be the number of vertices in  $\Delta$  and (re)label the vertices  $1, \dots, n$ . Let the board  $B$  be a disjoint union of  $n$  cycles of size 3 and 4 and label these  $1, \dots, n$  so that cycle  $i$  will have size 3 if the vertex  $i$  in  $\Delta$  belongs to  $\mathfrak{L}$ , and size 4 if the vertex  $i$  belongs to  $\mathfrak{R}$ .

Let  $G$  be the SP-game in which Left plays cycles of length 3, and Right plays cycles of length 4. Note that  $G$  is invariant.

It is easy to see that  $\Delta = \Delta_{G,B}$ .  $\square$

The following two examples demonstrate this construction in both the case where  $\Delta$  is not a simplex and when it is.

**Example 2.15.** Consider the complex  $\Delta = \langle ab, bc \rangle$ , where the vertices are partitioned as  $\mathfrak{L} = \{a, b\}$  and  $\mathfrak{R} = \{c\}$ . Since  $\Delta$  is not a simplex, we will follow the construction given in the first case of the proof of Theorem 2.14.

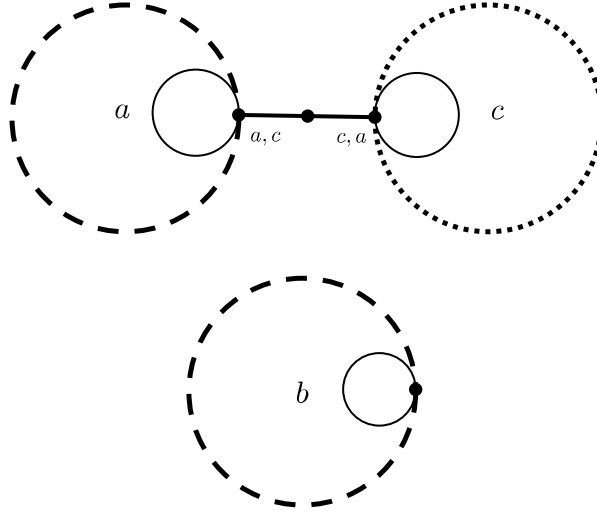
The only minimal nonface of  $\Delta$  is  $ac$ , thus the graph  $H$  is  $P_2$ . Since  $n = 2$ , in the SP-game  $G$  Left will play cycles of length  $n^4 + 4 = 20$  with one cycle of length  $n^3 = 8$  added to a vertex, while Right plays cycles of length  $n^4 + 5 = 21$  with a cycle of length 8 added to a vertex.

The board  $B$  is given in Fig. 8. Dashed cycles consist of 20 vertices, and dotted cycles of 21 vertices. The smaller solid cycles consist of 8 vertices.

It is now easy to check that  $\Delta_{G,B} = \Delta$ .

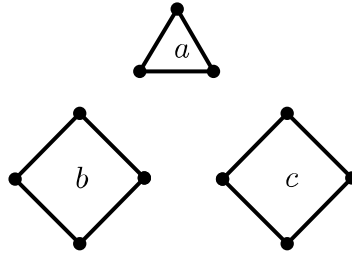
**Example 2.16.** Consider the simplex  $\Delta = \langle abc \rangle$ , where the vertices are partitioned as  $\mathfrak{L} = \{a\}$  and  $\mathfrak{R} = \{b, c\}$ . Since  $\Delta$  is a simplex, we will follow the second construction given in the proof of Theorem 2.14. Since  $n = 3$ , in



FIGURE 8. Constructing  $B$  from  $\Delta = \langle ab, bc \rangle$ 

the SP-game  $G$  Left will play cycles of length 3, while Right plays cycles of length 4.

The board  $B$  is given in Fig. 9.

FIGURE 9. Constructing  $B$  from  $\Delta = \langle abc \rangle$ 

It is now easy to check that  $\Delta_{G,B} = \Delta$  and that  $\Gamma_{G,B}$  is empty.

Concluding our discussion of iSP-games, we have the following result.

**Theorem 2.17 (Every SP-Game Tree Belongs To An iSP-Game).**  
*Given an SP-game  $G$  played on a board  $B$ , there exists an iSP-game  $G'$  played on a board  $B'$  so that their game trees are isomorphic.*

*Proof.* Let  $\Delta = \Delta_{G,B}$  with  $\mathfrak{L}$  the vertices corresponding to Left basic positions, and similarly  $\mathfrak{R}$ . Then by Theorem 2.14 we know that there exists an iSP-game  $G'$  and a board  $B'$  such that  $\Delta = \Delta_{G',B'}$  with the same bipartition. Since  $\Delta_{G,B} = \Delta_{G',B'}$ , we have by Proposition 1.13 that the game trees of  $G$  played on  $B$  and  $G'$  played on  $B'$  are isomorphic.  $\square$

This in particular implies that under most winning conditions (such as normal play or misère play) the game values of  $G$  played on  $B$  and  $G'$  played on  $B'$  are the same, implying that we can replace one by the other.

### 3. INDEPENDENCE GAMES

Many of the games we have previously considered have illegal complexes that are graphs. This special class of SP-games is of further interest to us. For example, this class corresponds to flag complexes (see below for more).

**Definition 3.1.** An SP-game  $G$  is called an **independence game** if for *any* board  $B$  the illegal complex  $\Gamma_{G,B}$  is a graph without isolated vertices (i.e. a pure one-dimensional simplicial complex).

Consider the illegal complex  $\Gamma_{G,B}$  of an independence game  $G$  on a board  $B$ . Let  $\Gamma'_{G,B}$  be the graph on the vertex set  $x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_n$  (corresponding to the basic positions of  $G$  played on  $B$ ) with edges those of  $\Gamma_{G,B}$ . Thus the difference between  $\Gamma_{G,B}$  and  $\Gamma'_{G,B}$  are isolated vertices corresponding to basic positions that are always legal. For many independence games we have  $\Gamma'_{G,B} = \Gamma_{G,B}$ .

The **independence complex** of a graph  $H$  is a simplicial complex with vertex set that of the graph and faces those sets of vertices that are independent in  $H$ , i.e. no two vertices are adjacent. The term ‘independence game’ was chosen for this class of games since the independence sets of  $\Gamma'_{G,B}$  correspond to the legal positions of  $G$  played on  $B$ , i.e. the faces of  $\Delta_{G,B}$ . Thus in this case  $\Delta_{G,B}$  is the independence complex of the graph  $\Gamma'_{G,B}$ .

Many SP-games, such as COL, SNORT, and all PARTIZAN OCTAL games are independence games. NOGO is an example of an SP-game that is not an independence game. Even though  $\Gamma_{\text{NoGo},B}$  is a graph for some boards (for example when  $B$  is the graph on two vertices connected by an edge, i.e. the path of length one,  $P_2$ ), there are many others for which this is not the case. For example,  $\Gamma_{\text{NoGo},P_3}$ , given in Fig. 10, has two-dimensional faces.

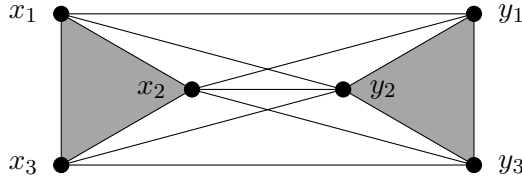


FIGURE 10. The Illegal Complex  $\Gamma_{\text{NoGo},P_3}$

One nice property of independence games is that playing an independence game  $G$  on a board  $B$  is equivalent to forming independence sets of the graph  $\Gamma'_{G,B}$  while Left picks vertices in  $\mathfrak{L}$  and Right in  $\mathfrak{R}$ .

A **flag complex**  $\Delta$  is a complex whose minimal non-faces all have size 2 (see for example [12]). In the case of independence games, since  $\Gamma_{G,B}$  is a graph without isolated vertices, we have that  $\Delta_{G,B}$  is flag.

Further note that the  $\Gamma$ -game in the case of  $\Gamma$  being a graph is always an independence game (since minimal illegal positions are always pairs of pieces played). Using Theorem 2.14 this implies the following.

**Proposition 3.2 (iSP-Games of Flag Complexes).** *Given any SP-game  $G$  and board  $B$  such that  $\Gamma_{G,B}$  is a non-empty graph, there exists an independence game  $G'$  and board  $B'$  such that  $\Delta_{G,B} = \Delta_{G',B'}$ . In the case that  $\Gamma_{G,B}$  has no isolated vertices, we also have  $\Gamma_{G,B} = \Gamma_{G',B'}$ .*

*Proof.* By Theorem 2.14 there exists an iSP-game  $G'$  and board  $B'$  such that  $\Delta_{G,B} = \Delta_{G',B'}$ . In the case that  $\Delta_{G,B}$  is not a simplex (if  $\Gamma_{G,B}$  has at least one edge), the game  $G'$  is the  $\Gamma$ -game  $G_{\Gamma_{G,B}}$ . As mentioned above, this is an independence game. In the case that  $\Delta_{G,B}$  is a simplex, the game  $G'$  has no illegal positions, and thus is an independence game trivially.

If  $\Gamma_{G,B}$  has no isolated vertices, then the underlying rings of  $\Delta_{G,B}$  and  $\Delta_{G',B'}$  are the same, thus

$$\Gamma_{G',B'} = \mathcal{F}(\mathcal{N}(\Delta_{G',B'})) = \mathcal{F}(\mathcal{N}(\Delta_{G,B})) = \Gamma_{G,B}. \quad \square$$

Equivalently, this proposition also states that given an SP-game  $G$  and board  $B$  such that the minimal non-faces of  $\Delta_{G,B}$  are all 1- and 2-element sets, there exists an SP-game  $G'$  whose legal complex is always flag and a board  $B'$  such that  $\Delta_{G,B} = \Delta_{G',B'}$ .

As a direct consequence of Proposition 3.2, applying Proposition 1.13, we have that these games also have isomorphic game trees.

**Corollary 3.3.** *Given any SP-game  $G$  and board  $B$  such that  $\Gamma_{G,B}$  is a non-empty graph, there exists an independence game  $G'$  and board  $B'$  such the game trees of  $(G, B)$  and  $(G', B')$  are isomorphic.*

#### 4. FURTHER QUESTIONS AND WORK

In this section, we will be discussing some potential further questions and avenues to explore.

The  $\Gamma$ -board and pieces of the  $\Gamma$ -game have many more vertices than  $\Gamma$  itself. Thus we are interested in whether constructions of a game  $G$  and board  $B$  are possible for every simplicial complex  $\Gamma$  without isolated vertices in which the pieces that Left and Right play occupy only one vertex so that  $\Gamma = \Gamma_{G,B}$ . This seems unlikely though, thus an interesting question is for which class of simplicial complexes such a construction is possible.

Similarly, we are also interested in for which simplicial complexes  $\Delta$  we can find a game  $G$  and board  $B$  with pieces only a single vertex so that  $\Delta = \Delta_{G,B}$ .

A flag complex  $\Delta$  is a complex whose minimal non-faces all have size 2. Thus if  $\Delta$  is flag, then  $\Gamma = \mathcal{F}(\mathcal{N}(\Delta))$  is a graph without isolated vertices. Simplicial trees and forests, which are generalizations of graph trees and forests, are flag complexes (see [12, Lemma 9.2.7]), and thus game complexes. Since many properties of simplicial trees are known (see for example [7] and [8]) it seems that this class of flag complexes provides a good start to studying whether simpler constructions are possible.

Finally, it is of interest if each game value possible under normal play conditions is also the game value of some SP-game. This problem has received attention for specific SP-games (for DOMINEERING see for example [14, 21], for COL and SNORT see [3]), and was recently positively answered for a non-SP-game (see [6]). Since SP-games are much easier to understand than many other combinatorial games, if the answer to this question is positive, it would provide an excellent new tool for studying combinatorial games. Whether or not this is the case, a similar, but stronger, question is if the simplest game (i.e. the game value) in each equivalence class containing an SP-game is itself an SP-game. Knowing that each simplicial complex is the legal complex of some SP-game has been indispensable in the exploration of those two questions (see [13]).

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